The Limit Cycles of Liénard Equations in the Weakly Nonlinear Regime

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Abstract

Liénard equations of the form $\ddot{x} + \epsilon f(x)\dot{x} + x = 0$, with $f(x)$ an even function, are considered in the weakly nonlinear regime ($\epsilon \to 0$). A perturbative algorithm for obtaining the number, amplitude and shape of the limit cycles of these systems is given. The validity of this algorithm is shown and several examples illustrating its application are given. In particular, an $\mathcal{O}(\epsilon^8)$ approximation for the amplitude of the van der Pol limit cycle is explicitly obtained.

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1 Introduction

A pattern formed by a continuum of periodic orbits is the usual look of phase space for a conservative planar system. Take, for instance, the harmonic oscillator $\ddot{x} + x = 0$. The set of circles $x^2 + \dot{x}^2 = a^2$, with a a positive real number representing the amplitude of the oscillation, fills the whole plane $(x, \dot{x})$. Nevertheless, dissipation and amplification are always present in real dynamics, since the oscillating system is usually embedded in an interacting external medium. The effect of this external medium is implemented in the evolution equations of the oscillator, with more or less accuracy, through a nonlinear term $h(x, \dot{x})$ that depends on the position and velocity of the oscillator. In general, even if this new term is only a slight perturbation governed by the small real parameter ϵ, the coexistence of the infinitely many periodic motions of the original unperturbed system is destroyed and only a finite number of them survives in the new context given by the equation $\ddot{x} + x + \epsilon h(x, \dot{x}) = 0$. Dissipation and amplification are balanced on those orbits. These localized periodic motions that verify strict energetic balance conditions are called *self-sustained oscillations* or *limit cycles* [1, 2].

Liénard equations,

$$\ddot{x} + \epsilon f(x)\dot{x} + x = 0, \tag{1}$$

with $h(x, \dot{x}) = f(x)\dot{x}$ and $f(x)$ any real function, model the dynamics of specific planar systems where limit cycles can be found. When $f(x)$ is a polynomial of degree $N = 2n+1$ or $2n$, with n a natural number, Lins, Melo and Pugh have conjectured (LMP-conjecture) that the maximum number of limit cycles allowed is just n [3]. It is true if $N = 2$, or $N = 3$ or if $f(x)$ is even and $N = 4$ [3, 4]. Also, it has been claimed its truth in the strongly nonlinear regime ($\epsilon \to \infty$) when $f(x)$ is an even polynomial [5]. There are no general results about the limit cycles when $f(x)$ is a polynomial of degree greater than 5 neither, in general, when $f(x)$ is an arbitrary real function [6]. When $f(x) = x^2 - 1$ we have the van der Pol oscillator, $\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0$, wich is a particular example of Liénard system that has been very well studied. It displays a limit cycle whose uniqueness and non-algebraicity has been shown for the whole range of the parameter ϵ [7]. Its behavior

runs from near-harmonic oscillations when $\epsilon \to 0$ to relaxation oscillations when $\epsilon \to \infty$, making it a good model for many practical situations [8].

Different non-perturbative approaches that allow to obtain information about the number of limit cycles and their location in phase space have been proposed in the last years when $f(x)$ is an even polynomial. A method that gives a sequence of algebraic approximations to the equation of each limit cycle can be found in [6], and a variational method showing that limit cycles correspond to relative extrema of certain functionals is explained in [9].

In this paper, we are interested in determining the number, amplitude and shape of the limit cycles emerging in the weakly nonlinear regime ($\epsilon \to 0$) of Liénard equations by standard perturbation techniques. This work completes a previous work [5] in which these questions were successfully answered and solved in the strongly nonlinear regime ($\epsilon \to \infty$). In fact, the algorithm here explained is an alternative to a more general, although technically more sophisticated, perturbative method proposed by Giacomini et al. [10, 11] on the same question. Thus, we propose a new algorithm which permits an easy computation up to an arbitrary order $\mathcal{O}(\epsilon^N)$ of the amplitude a and form $y(x)$ of the limit cycles. Moreover, this simple algorithm permits to show some interesting properties of symmetry of the approximate solutions. The article is organized as follows. A few symmetries of the Liénard equation are recalled in Section 2. The method is detailed in Section 3. Some illustrative examples are given in Section 4. Finally, we present our conclusions.

2 Symmetries and an integral equation

In order to study the limit cycles of equation (1) with the time variable being implicit, it is convenient to rewrite it in the coordinates $(x, \dot{x}) = (x, y)$ in the plane, with $\dot{x}(t) = y(x)$ and $\ddot{x}(t) = y(x)y'(x)$ (where $y'(x) = dy/dx$):

$$yy' + \epsilon f(x)y + x = 0. \tag{2}$$

A limit cycle $C_l \equiv (x, y_\pm(x))$ of equation (2) has a positive branch $y_+(x) > 0$ and a negative branch $y_-(x) < 0$. They cut the x-axis in two points $(-a_-, 0)$ and $(a_+, 0)$ with $a_-, a_+ > 0$ because the origin $(0,0)$ is the only fixed point of Eq. (2). Then every limit cycle C_l solution of Eq. (2) encloses the origin and the oscillation x runs in the interval $-a_- < x < a_+$. The amplitudes of oscillation a_-, a_+ identify the limit cycle. The result is a nested set of closed curves that defines the qualitative distribution of the integral curves in the plane (x, y). The stability of the limit cycles is alternate. For a given stable limit cycle, the two neighboring limit cycles, the closest one in its interior and the closest one in its exterior, are unstable, and vice versa.

When $f(x)$ is an even function, the *inversion symmetry* $(x, y) \leftrightarrow (-x, -y)$ of Eq. (2) implies $y_+(x) = -y_-(-x)$ and then $a_1 = a_2 = a$. Therefore we can restrict ourselves to the positive branches of the limit cycles $(x, y_+(x))$ with $-a \le x \le a$. In this case, the amplitude a identifies the limit cycle. The *parameter inversion symmetry* $(\epsilon, x, y) \leftrightarrow (-\epsilon, x, -y)$ implies that if $C_l \equiv (x, y_\pm(x))$ is a limit cycle for a given ϵ, then $\overline{C_l} \equiv (x, -y_\mp(x))$ is a limit cycle for $-\epsilon$. In consequence, the amplitude a of the limit cycles in Liénard systems is an even function of ϵ. Moreover if C_l is stable (or unstable) then $\overline{C_l}$ is unstable (or stable, respectively). Therefore it is enough to consider the limit cycles when $\epsilon > 0$ for obtaining all of the periodic solutions. (The limit cycles for a given $\epsilon < 0$ are obtained from a reflection over the x-axis of those limit cycles obtained for $\epsilon > 0$).

Another property of a limit cycle can be derived from the fact that the mechanical energy $E = (x^2 + y^2)/2$ is conserved in an oscillation:

$$\int_{C_l} \frac{dE}{dx} dx = 0.$$

Taking into account that $dE = (yy' + x)\, dx = -\epsilon f(x) y\, dx$, if equation (2) is integrated along the positive branch of a limit cycle, between the maximal amplitudes of oscillation, we obtain:

$$\int_{-a}^{a} f(x) y_+(x) dx = 0. \tag{3}$$

The solutions $y_+(x)$ of Eqs. (2) and (3) and vanishing in the extremes, constitute the finite set of limit cycles of Eq. (2).

3 The perturbative method

For later convenience, we rewrite the Liénard equation (2) with a different notation:

$$uu' + \epsilon f(t)u + t = 0, \tag{4}$$

where $f(t)$ is an even function of the real variable t and u' means derivative with respect to t. From the discussion of the above section we know that the amplitudes of the limit cycles are symmetric: if $u(t)$ is a limit cycle of amplitude a then $u(-a) = u(a) = 0$. After the following change of dependent and independent variables:

$$t = ax, \qquad u = ay, \qquad a > 0, \tag{5}$$

equation (4) reads

$$yy' + \epsilon f(ax)y + x = 0, \tag{6}$$

where a is now an explicit positive parameter of the equation, and y' means now derivative with respect to x. Every limit cycle $u(t)$ of (4) of amplitude a is transformed into a limit cycle solution $y(x)$ of (6) of amplitude 1 verifying $y(-1) = y(1) = 0$. The main result of this paper is to establish in these new variables (x, y) a recursive algorithm capable of approximating a limit cycle solution $y(x)$ and its amplitude a in a power series of ϵ up to an arbitrary order. The approximated limit cycles in the original variables (u, t) are obtained by undoing the change of variables (5) at each order $\mathcal{O}(\epsilon^N)$ of approximation. In order to show our results we need some previous definitions.

Definition 1. For $N = 0, 1, 2, \ldots$, we denote by $y^{(N)}(x)$ the function that approximates a limit cycle solution $y(x)$ of (6) up to the order $\mathcal{O}(\epsilon^N)$:

$$y^{(N)}(x) \equiv \sum_{n=0}^{N} \epsilon^n y_n(x). \tag{7}$$

We denote by $a^{(N)}$ the approximation to its amplitude a up to the order $\mathcal{O}(\epsilon^N)$:

$$a^{(N)} \equiv \sum_{n=0}^{N-1} a_n \epsilon^n. \tag{8}$$

The coefficient functions $y_n(x)$ and the coefficients a_n are computed by means of the algorithm given below.

Definition 2. For $n = 0, 1, 2, ..., N$, we denote by $f_n(a_0, a_1, ..., a_n; x)$ the coefficients of the formal Taylor expansion of the function $g(\epsilon) \equiv f(a^{(N)}x)$ at $\epsilon = 0$:

$$f\left(x \sum_{n=0}^{N} a_n \epsilon^n\right) = \sum_{n=0}^{N} f_n(a_0, a_1, ..., a_n; x)\epsilon^n + \mathcal{O}(\epsilon^{N+1}).$$

The first few coefficients $f_n(a_0, a_1, ..., a_n; x)$ are:

$$f_0(a_0; x) = f(a_0 x),$$

$$f_1(a_0, a_1; x) = x f'(a_0 x) a_1,$$

$$f_2(a_0, a_1, a_2; x) = x f'(a_0 x) a_2 + \frac{1}{2} x^2 f''(a_0 x) a_1^2,$$

$$f_3(a_0, a_1, a_2, a_3; x) = x f'(a_0 x) a_3 + x^2 f''(a_0 x) a_1 a_2 + \frac{1}{6} x^3 f'''(a_0 x) a_1^3.$$

The general form of the coefficients $f_n(a_0, a_1, ..., a_n; x)$ is given in the following lemma whose proof is straightforward and we do not reproduce here:

Lemma 1. For $n = 0$ we have $f_0(a_0; x) = f(a_0 x)$ and, for $n = 1, 2, ...,$

$$f_n(a_0, a_1, ..., a_n; x) = \sum_{m=1}^{n} b_{n,m}(a_1, ..., a_n) x^m f^{(m)}(a_0 x), \tag{9}$$

where

$$b_{n,m}(a_1, ..., a_n) \equiv \sum_{S_{n,m}} \frac{a_1^{\sigma(1)} a_2^{\sigma(2)} \cdots a_n^{\sigma(n)}}{\sigma(1)!\sigma(2)! \cdots \sigma(n)!}.$$

In these formulas $n \geq m \geq 1$ and the above sum is performed over the following subset $S_{n,m}$ of the set of all of the possible mappings σ between the sets $\{1, 2, 3, ..., n\}$ and $\{0, 1, 2, 3, ..., m\}$:

$$S_{n,m} \equiv \left\{ \sigma : \{1, 2, 3, ..., n\} \to \{0, 1, 2, 3, ..., m\};\ \sum_{k=1}^{n} \sigma(k) = m,\ \sum_{k=1}^{n} k\sigma(k) = n \right\}.$$

We also define $b_{0,0} = 1$ and $b_{n,0} = 0$ for $n > 0$.

Definition 3. Suppose that $y_0(x)$, $y_1(x)$,..., $y_n(x)$ are given. For $n = 0, 1, 2, ...,$ we define the functions:

$$\begin{aligned} \beta_n(a_0, a_1, ..., a_n) &\equiv \sum_{m=0}^{n} \int_{-1}^{1} f_m(a_0, a_1, ..., a_m; x) y_{n-m}(x) dx = \\ &= \sum_{m=0}^{n} \sum_{k=0}^{m} b_{m,k} \int_{-1}^{1} x^k y_{n-m}(x) f^{(k)}(a_0 x) dx. \end{aligned} \tag{10}$$

Algorithm. *We compute the coefficients* (y_n, a_n), $n = 0, 1, 2, ..., N$, *of expansions (7) and (8) in the following way.*

<u>**Step 1**</u>*: Set* $y_0(x) = \sqrt{1 - x^2}$ *and take one of the solutions* a_0 *of the (in general nonlinear) equation* $\beta_0(a_0) = 0$*:*

$$\beta_0(a_0) \equiv \int_{-1}^{1} f_0(a_0 x) y_0(x) dx = 0. \tag{11}$$

<u>**Step 2**</u>*: For each* n*, with* $n = 1, 2, ..., N$*, compute, firstly,* $y_n(x)$ *by means of the formula*

$$\begin{aligned} y_n(x) &= -\frac{1}{y_0(x)} \left\{ \frac{1}{2} \sum_{m=1}^{n-1} y_m(x) y_{n-m}(x) + \sum_{m=0}^{n-1} \int_{-1}^{x} y_{n-m-1}(t) f_m(a_0, a_1, ..., a_m; t) dt \right\} = \\ &= -\frac{1}{y_0(x)} \left\{ \frac{1}{2} \sum_{m=1}^{n-1} y_m(x) y_{n-m}(x) + \sum_{m=0}^{n-1} \sum_{k=0}^{m} b_{m,k} \int_{-1}^{x} t^k y_{n-m-1}(t) f^{(k)}(a_0 t) dt \right\} \end{aligned} \tag{12}$$

and, secondly, solve for a_n *the linear equation*

$$\beta_n(a_0, a_1, ..., a_n) = 0. \tag{13}$$

The derivation of this algorithm is given in the following theorem. Its detailed form for $N = 5$ is given in Section 4.

Theorem 1. For $N = 0, 1, 2, ...$, the function $y^{(N)}(x)$ and the amplitude $a^{(N)}$ obtained by means of the above algorithm satisfy (6) up to the order $\mathcal{O}(\epsilon^{N+1})$ uniformly in $x \in [-1, 1]$:

$$y^{(N)} \frac{dy^{(N)}}{dx} + \epsilon f(a^{(N)} x) y^{(N)} + x = \mathcal{O}(\epsilon^{N+1}). \tag{14}$$

Moreover:

(i) All of the coefficients $y_n(x)$ of $y^{(N)}(x)$ are continuous functions of x and satisfy $y_n(-1) = y_n(1) = 0$ and therefore $y^{(N)}(-1) = y^{(N)}(1) = 0 \ \forall \ N$.

(ii) If $\beta_0'(a_0) \neq 0$ (limit cycles of multiplicity higher than one are not considered here) then $a_{2n+1} = f_{2n+1}(a_0, a_1, ..., a_{2n+1}; x) = 0$ for $n = 0, 1, 2, ...$ The coefficients $y_{2n+1}(x)$ are odd functions of x and $y_{2n}(x)$ are even functions of x.

(iii) When $f(x)$ is a polynomial of degree q in x^2, then

$$y_{2n}(x) = p_{2n}(x^2)(1-x^2)^{3/2}, \qquad n = 1, 2, 3, ...,$$

$$y_{2n+1}(x) = x p_{2n+1}(x^2)(1-x^2), \qquad n = 0, 1, 2, ...,$$

where $p_n(x)$ is a polynomial of degree $nq - 1$.

Proof. If we replace y by $y^{(N)}$ and a by $a^{(N)}$ in (6) and equate powers of ϵ we obtain that

$$y_0 y_0' + x = 0 \tag{15}$$

and, for $n = 1, 2, 3, ...$, every function $y_n(x)$ satisfies a first order linear differential equation which contains y_0, y_1,...,y_{n-1}:

$$\frac{1}{2}\sum_{m=0}^{n}(y_m y_{n-m})' + \sum_{m=0}^{n-1} f_m(a_0, a_1, ..., a_m; x) y_{n-m-1} = 0. \tag{16}$$

The solution of (15) which satisfies $y_0(-1) = 0$ is obviously $y_0(x) = \sqrt{1-x^2}$. We solve recursively every one of the equations (16) for $y_n(x)$, $n = 1, 2, ..., N$, in terms of the preceding functions y_0, y_1,..., y_{n-1} and of the coefficients a_0, a_1,..., a_{n-1} by imposing that $y_n(-1) = 0$ for $n = 1, 2, 3, ..., N$. Then we obtain (12). Therefore, $y^{(N)}$ satisfies (6) with a replaced by $a^{(N)}$ up to the order $\mathcal{O}(\epsilon^{N+1})$ and (14) holds.

We already have that $y_n(-1) = 0$ for $n = 0, 1, 2, ..., N$. To show thesis (i), it remains to show that $y_n(1) = 0$ for $n = 0, 1, 2, ..., N$ when Eq. (13) is assumed. We write (12) in the form

$$y_n(x) = w_n(x) - \frac{\beta_{n-1}(a_0, a_1, ..., a_{n-1})}{y_0(x)}, \tag{17}$$

with

$$w_n(x) \equiv -\frac{1}{y_0(x)}\left\{\frac{1}{2}\sum_{m=1}^{n-1} y_m(x) y_{n-m}(x) + \sum_{m=0}^{n-1}\sum_{k=0}^{m} b_{m,k}\int_x^1 t^k y_{n-m-1}(t) f^{(k)}(a_0 t) dt\right\} \tag{18}$$

and $\beta_n(a_0, a_1, ..., a_n)$ given in (10). Obviously, $y_0(x) = \mathcal{O}(\sqrt{1-x})$ when $x \to 1$ and is a continuous function in $[-1, 1]$. From the hypothesis (13), we have that $\beta_n(a_0, a_1, ..., a_n) = 0$ for $n = 0, 1, 2, ..., N-1$ and then $y_n(x) = w_n(x)$ for $n = 1, 2, 3, ..., N$. From (18), it is straightforward to show by induction over n that all of the functions $w_n(x)$ are continuous

in $[-1,1]$ and $w_n(x) = \mathcal{O}(\sqrt{1-x})$ when $x \to 1$. Therefore, $y_n(x)$ is continuous in $[-1,1]$ and $y_n(1) = 0$ for $n = 0, 1, 2, ..., N$. Then thesis (i) holds.

From Definition 2, all of the odd functions $f_{2n+1}(a_0, a_1, ..., a_{2n+1}; x)$ vanish if the odd coefficients a_{2n+1} vanish. This is so because it is necessary to have a couple $(k, \sigma(k))$ with odd k and $\sigma(k)$ in the definition of $b_{n,m}$ in order to satisfy the condition $\sum_{k=1}^{2n+1} k\sigma(k) = 2n+1$. Then, to show the first part of (ii) it suffices to show that $a_{2n+1} = 0$, with $n = 0, 1, 2, ...$

The function $y_0(x)$ is even and from (12) with $n = 1$:

$$y_1(x) = -\frac{1}{y_0(x)} \int_{-1}^{x} f(a_0 t) y_0(t) dt. \tag{19}$$

Taking into account that $\beta_0(a_0) = \int_{-1}^{1} f(a_0 t) y_0(t) dt = 0$ and that $f(x)$ is even, we have $\int_{-1}^{0} f(a_0 t) y_0(t) dt = \frac{1}{2}\beta_0(a_0) = 0$. We can write the integral in (19) in the form $\int_{-1}^{x} f(a_0 t) y_0(t) dt = \int_{-1}^{0} f(a_0 t) y_0(t) dt + \int_{0}^{x} f(a_0 t) y_0(t) dt = \int_{0}^{x} f(a_0 t) y_0(t) dt$, which shows that $y_1(x)$ is odd. From (10) with $n = 1$ we have

$$\beta_1(a_0, a_1) = \int_{-1}^{1} f(a_0 x) y_1(x) dx + a_1 \int_{-1}^{1} x f'(a_0 x) y_0(x) dx.$$

The first integral vanishes because its integrand is odd and the second one equals $\beta_0'(a_0)$, which is not zero from assumption. Therefore, $\beta_1(a_0, a_1) = 0 \Rightarrow a_1 = 0$ and $f_1(a_0, a_1; x) = 0$. Then, thesis (ii) is true for $n = 0$. Suppose that it is true for $n = 0, 1, 2, ..., m-1$ with $m > 0$; we will show that thesis (ii) is also true for $n = m$. If thesis (ii) is true for $n = 0, 1, 2, ..., m-1$, then for $n \le m$, the integrals in the first line of (12) may be written in the form

$$\begin{aligned}
&\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \int_{-1}^{x} y_{n-2k-1}(t) f_{2k}(a_0, a_1, ..., a_{2k}; t) dt = \\
&= \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \int_{-1}^{0} y_{n-2k-1}(t) f_{2k}(a_0, a_1, ..., a_{2k}; t) dt + \\
&\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \int_{0}^{x} y_{n-2k-1}(t) f_{2k}(a_0, a_1, ..., a_{2k}; t) dt.
\end{aligned} \tag{20}$$

If $f(x)$ is an even function of x, then all of the functions $f_{2k}(a_0, a_1, ..., a_{2k}; x)$ are even. When n is even then $y_{n-2k-1}(t)$, $k = 0, 1, 2, ..., \lfloor (n-1)/2 \rfloor$, is odd and the function

$\int_{-1}^{x} y_{n-2k-1}(t) f_{2k}(a_0, a_1, ..., a_{2k}; t)dt$ are even functions of x. On the other hand, when n is odd then $y_{n-2k-1}(t)$, $k = 0, 1, 2, ..., \lfloor (n-1)/2 \rfloor$, are even and the first sum in the right hand side of (20) equals $\beta_{n-1}(a_0, a_1, ..., a_{n-1})/2 = 0$ and then the left hand side is an odd function of x. Obviously, the sum $\sum_{k=1}^{n-1} y_k y_{n-k}$ in (12) is an even function of x if $n \le m$ is even and an odd function of x if $n \le m$ is odd. Then, the second part of thesis (ii) is true for $n = m$.

From the induction hypothesis, $a_{2n+1} = f_{2n+1} = 0$ for $n = 0, 1, 2, ..., \lfloor m/2 \rfloor - 1$ and then, for odd $n = m$, equation (10) reads:

$$\beta_m(a_0, a_1, ..., a_m) = \int_{-1}^{1} f_m(a_0, a_1, ..., a_m; x) y_0(x) dx. \tag{21}$$

From Definition 2, for odd m, the function $f_m(a_0, a_1, ..., a_m; x)$ is of the form $f_m(a_0, a_1, ..., a_m; x) = x f'(a_0 x) a_m + g_m(a_1, a_3, ..., a_{m-2}; x)$, where $g_m(a_1, a_3, ..., a_{m-2}; x) = 0$ when $a_1 = a_3 = ... = a_{m-2} = 0$. Then, for odd $n = m$ equation (21) may be written in the form

$$\beta_m(a_0, a_1, ..., a_m) = a_m \beta_0'(a_0).$$

Therefore, the equation $\beta_m(a_0, a_1, ..., a_m) = 0 \Rightarrow a_m = 0$ (and then $f_m = 0$) for odd m. Then the first part of thesis (ii) is true for $n = m$.

To show (iii) we recall the function y_1 given in (19). If $f(x)$ is a polynomial of degree q in x^2, then the integrand involved in the computation of $y_1(x)$ is a product of a polynomial of degree q in t^2 by $\sqrt{1-t^2}$. An easy computation shows that

$$\int_{-1}^{x} f(a_0 t)\sqrt{1-t^2} = \alpha + x p(x^2)\sqrt{1-x^2} + \beta \arcsin(x).$$

where α and β are real numbers and $p(x)$ a polynomial of degree q. But $y_1(\pm 1) = 0$ and then

$$\int_{-1}^{1} f(a_0 t)\sqrt{1-t^2} = \alpha \pm \beta \arcsin(1) = 0 \Rightarrow \alpha = \beta = 0.$$

Moreover, $y_1(x)$ is odd and from (19) and $\beta_0(a_0) = 0$, $y_1(x) = \mathcal{O}(1-x^2)$ when $x \to \pm 1$. These facts mean that $y_1(x) = x p_1(x^2)(1-x^2)$, where $p_1(x)$ is a polynomial of degree $q-1$. From these arguments and using induction over n in formula (12) we can easily obtain thesis (iii).

4 The first few terms of the expansion

For $\epsilon = 0$, Eq. (4) has a continuum of periodic solutions: $u(t) = ay_0(t/a)$ for any value of a. Hence, $y_0(1) = 0$ does not impose any restriction over the parameter a. we determine $a_0, ..., a_{N-1}$ and $y_1(x)$,...,$y_N(x)$ by means of the above introduced algorithm. First, we take $y_0(x) = \sqrt{1-x^2}$. Then, for $n = 0, 1, 2, ..., N-1$, solve the equation $\beta_n(a_0, a_1, ..., a_n) = 0$ for a_n and then compute $y_{n+1}(x)$. We give below the details of this algorithm for $N = 5$.

Definition 4. For convenience we define, for $n = 2, 4, 6, ...$, the integrals

$$\gamma_n(a_0, a_2, ..., a_{n-2}) \equiv \int_{-1}^{1} f(a_0 x) y_n(x) dx.$$

<u>The order $n = 0$</u>. We calculate a_0 and $y_1(x)$ as follows. Compute the positive a_0 solutions of

$$\beta_0(a_0) = \int_{-1}^{1} f(a_0 x)\sqrt{1-x^2} dx = 0.$$

From (12) we obtain the expression (19) for $y_1(x)$:

$$y_1(x) = -\frac{1}{y_0(x)} \int_{-1}^{x} f(a_0 t) y_0(t) dt.$$

<u>The order $n = 1$</u>. We know that $a_1 = 0$ and we calculate $y_2(x)$ from (12) with $n = 2$:

$$y_2(x) = -\frac{1}{y_0(x)} \left[\frac{1}{2} y_1^2(x) + \int_{-1}^{x} f(a_0 t) y_1(t) dt\right].$$

<u>The order $n = 2$</u>. We calculate a_2 and $y_3(x)$. From (10) with $n = 2$ we have

$$\beta_2(a_0, 0, a_2) = \int_{-1}^{1} f(a_0 x) y_2(x) dx + a_2 \int_{-1}^{1} x f'(a_0 x) y_0(x) dx = \gamma_2(a_0) + a_2 \beta_0'(a_0) = 0$$

Then,

$$a_2 = -\frac{\gamma_2(a_0)}{\beta_0'(a_0)}. \tag{22}$$

From (12) with $n = 3$:

$$y_3(x) = -\frac{1}{y_0(x)} \left[y_1(x) y_2(x) + \int_{-1}^{x} f(a_0 t) y_2(t) dt + a_2 \int_{-1}^{x} t f'(a_0 t) y_0(t) dt\right].$$

<u>The order $n = 3$</u>. We know that $a_3 = 0$ and we calculate $y_4(x)$ from (12) with $n = 4$:

$$y_4(x) = -\frac{1}{y_0(x)}\left[y_1(x)y_3(x) + \frac{1}{2}y_2^2(x) + \int_{-1}^{x} f(a_0 t)y_3(t)dt + a_2\int_{-1}^{x} tf'(a_0 t)y_1(t)dt\right].$$

<u>The order $n = 4$</u>. We calculate a_4 and $y_5(x)$. From (10) with $n = 4$ we have

$$\beta_4(a_0, 0, a_2, 0, a_4) = \gamma_4(a_0, a_2) + a_2\gamma_2'(a_0) + \frac{a_2^2}{2}\beta_0''(a_0) + a_4\beta_0'(a_0) = 0$$

Then,

$$a_4 = -\frac{\gamma_4(a_0, a_2) + a_2\gamma_2'(a_0) + \frac{a_2^2}{2}\beta_0''(a_0)}{\beta_0'(a_0)}. \tag{23}$$

From (12) with $n = 5$:

$$\begin{aligned} y_5(x) &= -\frac{1}{y_0(x)}\left[y_1(x)y_4(x) + y_2(x)y_3(x) + \int_{-1}^{x} f(a_0 t)y_4(t)dt + a_2\int_{-1}^{x} tf'(a_0 t)y_2(t)dt + \right. \\ &\qquad \left. a_4\int_{-1}^{x} tf'(a_0 t)y_0(t)dt + \frac{a_2^2}{2}\int_{-1}^{x} t^2 f''(a_0 t)y_0(t)dt\right]. \end{aligned} \tag{24}$$

Remark 1. The integral $\beta_0(a_0)$ coincides with the first Melnikov function or, equivalently, with the Abelian integral [12] defined for the perturbed Hamiltonian system (6) whose level curves at $\epsilon = 0$ are given by the set of circles $x^2 + y^2 = a^2$. Thus, the equation $\beta_0(a_0) = 0$ is a nonlinear equation for a_0 with none, one or several solutions, whereas the remaining equations $\beta_n(a_0, a_1, ..., a_n) = 0$ with $n = 1, 2, .., N-1$ are linear equations in a_n with a unique solution for every a_0 solution of $\beta_0(a_0) = 0$. Then, the number of positive solutions a_0 of $\beta_0(a_0) = 0$ determine the maximal number of limit cycles emerging from the period annulus when it is perturbed and the $\mathcal{O}(1)$ approximation to their amplitude. The remaining equations $\beta_n(a_0, a_1, ..., a_n) = 0$ with $n = 1, 2, .., N-1$ determine recursively the coefficients a_1,...,a_{N-1}, which are the pertubative correction to the first order amplitudes a_0. In this way, for every solution of $\beta_0(a_0) = 0$, we completely determine the amplitude a and the form $y(x)$ of the limit cycle solutions of (6) up to the order $\mathcal{O}(\epsilon^N)$:

$$a \simeq a^{(N)} \equiv \sum_{n=0}^{N-1} a_n\epsilon^n, \qquad y(x) \simeq y^{(N)}(x) \equiv \sum_{n=0}^{N} \epsilon^n y_n(x).$$

If $f(x)$ is a polynomial of degree q in x^2, then $\beta_0(a_0)$ is a polinomial in a_0^2 of degree q. Then, for small ϵ, the maximum possible number of limit cycles of (6) is q, as it has been conjectured by Lins, Melo and Pugh (LMP-conjecture) [3].

Remark 2. The approximate solution $y^{(N)}(x)$ with amplitude $a^{(N)}$ has the same symmetries than the exact solutions of Eq. (6): the *inversion symmetry* $(x, y) \leftrightarrow (-x, -y)$ and the *parameter inversion symmetry* $(\epsilon, x, y) \leftrightarrow (-\epsilon, -x, y)$. These symmetries are not destroyed at the perturbative level.

Remark 3. From thesis (iii) of Theorem 1, when $f(x)$ is a polynomial of degree q in x^2 we have that

$$y^{(N)}(x) = \sqrt{1-x^2} + x(1-x^2) \sum_{n=0}^{\lfloor N/2 \rfloor} p_{2n+1}(x^2)\epsilon^{2n+1} + (1-x^2)^{3/2} \sum_{n=1}^{\lfloor N/2 \rfloor} p_{2n}(x^2)\epsilon^{2n},$$

where $p_n(x)$ is a polynomial of degree $nq - 1$.

5 Examples

We perform the calculations proposed in Section 4 for two concrete examples, which are particular cases of the families 1 and 3 worked out in [13].

Example 1. The van der Pol oscillator is given for $f(x) = x^2 - 1$. This system has a unique limit cycle, which is stable for $\epsilon > 0$. Hence, the only root of $\beta_0(a)$ is $a_0 = 2$. For this value of a_0, an approximate form of the limit cycle for small ϵ is $y^{(5)}(x) = \sum_{n=0}^{5} \epsilon^n y_n(x)$ with

$$y_0(x) = \sqrt{1-x^2}, \qquad y_1(x) = x(1-x^2),$$

$$y_2(x) = \frac{1}{12}(1+2x^2)(1-x^2)^{3/2}, \qquad y_3(x) = \frac{1}{72}x^3(6x^2-5)(1-x^2),$$

$$y_4(x) = \frac{1}{4320}(-4 - 23x^2 + 213x^4 - 156x^6)(1-x^2)^{3/2}$$

and

$$y_5(x) = \frac{1}{1036800}x^3(10385 - 43794x^2 + 41424x^4 - 10560x^6)(1-x^2).$$

An approximate form of its amplitude for small ϵ is $a(\epsilon) = a^{(6)} + \mathcal{O}(\epsilon^8)$, with

$$a^{(6)} = 2 + \frac{1}{96}\epsilon^2 - \frac{1033}{552960}\epsilon^4 + \frac{1019689}{55738368000}\epsilon^6. \tag{25}$$

We integrate Eq. (2) by a Runge-Kutta method in order to obtain the limit cycle. This curve is plotted in a continuous trace in Fig. 1(a-b) for $\epsilon = 0.8$ and $\epsilon = 2$, respectively. The approximated limit cycle $u^{(5)}(x) = a^{(5)} y^{(5)}(x/a^{(5)})$ is also plotted in those figures with a discontinuous trace for the same values of ϵ. Let us remark that, in this case, even up to $\epsilon = 3$, the approximation $u^{(5)}(x)$ to the limit cycle is very good. Our calculation (25) agrees with the computational calculation of the 'exact' amplitudes given in Table 1, and also with the calculations on this system presented in [10].

Example 2. The same process is performed for $f(x) = 5x^4 - 9x^2 + 1$. In this case, the system has two limit cycles, one stable and the other unstable. The polynomial $\beta_0(a)$ has two positive roots: $a_0 = \sqrt{\frac{9-\sqrt{41}}{5}} = 0.720677$ (unstable limit cycle for $\epsilon > 0$) and $\bar{a}_0 = \sqrt{\frac{9+\sqrt{41}}{5}} = 1.75517$ (stable limit cycle for $\epsilon > 0$).

For the first limit cycle we have $y^{(3)}(x) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \epsilon^3 y_3(x)$ with

$$y_0(x) = \sqrt{1-x^2},$$

$$y_1(x) = x(x^2-1)\left(1 + \frac{9\sqrt{41}-61}{15}x^2\right),$$

$$y_2(x) = (1-x^2)^{3/2}(0.06586 + 0.13173x^2 - 0.05241x^4 + 0.00505x^6)$$

and

$$y_3(x) = x^3(1-x^2)(0.06081 - 0.11246x^2 + 0.05384x^4 - 0.00999x^6 + 0.00006x^8).$$

An approximate expression of its amplitude for small ϵ is

$$a(\epsilon) = 0.720677 + 0.00390888\ \epsilon^2 + \mathcal{O}(\epsilon^4).$$

For the second limit cycle we have $y^{(3)}(x) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \epsilon^3 y_3(x)$ with

$$y_0(x) = \sqrt{1-x^2},$$

$$y_1(x) = x(x^2-1)\left(1 - \frac{9\sqrt{41}+61}{15}x^2\right),$$

$$y_2(x) = (1-x^2)^{3/2}(0.98791 + 1.97583x^2 - 2.71374x^4 + 6.2545x^6)$$

and

$$y_3(x) = x^3(1-x^2)(0.846907 - 2.30045x^2 - 1.11829x^4 - 25.2371x^6 + 28.2651x^8).$$

An approximate form of its amplitude for small ϵ is

$$a(\epsilon) = 1.75517 - 0.0178803\ \epsilon^2 + \mathcal{O}(\epsilon^4).$$

The comparison between the 'exact' and the approximated limit cycles can be seen in the plots of Fig. 2(a-b) for $\epsilon = 0.8$ and $\epsilon = 2$ and Fig. 3(a-b) for $\epsilon = 0.4$ and $\epsilon = 0.8$.

6 Conclusions

Periodic self-oscillations can arise in nonlinear systems. These are represented by isolated closed curves in phase space that we call limit cycles. The knowledge of the number, amplitude and shape of these solutions in a general nonlinear equation is an unsolved problem.

In this work, a recursive algorithm to approximate the form $y(x)$ and the amplitude a of limit cycles in the weakly nonlinear regime ($\epsilon \to 0$) of Liénard equations, $\dot{x} = y$, $\dot{y} + \epsilon f(x) y + x = 0$, with $f(x)$ an even function, has been presented. The symmetries of the exact limit cycles are maintained at the perturbative level. Several examples showing the application of this scheme have been given, and the accuracy of the method in comparison with the direct numerical integration has also been tested. In particular, we have explicitly obtained the $\mathcal{O}(\epsilon^8)$ approximation for the amplitude of the van der Pol limit cycle.

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Table 1. The value a_T represents the approximated amplitude $a(\epsilon)$ of the van der Pol limit cycle obtained from (25) for the indicated values of ϵ. The value a_E represents the amplitude a obtained by integrating directly the system with a Runge-Kutta method.

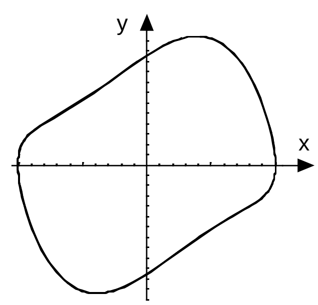
ϵ	0.1	0.2	0.3	0.4	0.5
a_T	2.000104	2.000414	2.000922	2.001619	2.002488
a_E	2.00010	2.00041	2.00092	2.00161	2.00248
ϵ	0.6	0.7	0.8	0.9	1
a_T	2.003509	2.004658	2.005906	2.007222	2.008569
a_E	2.00351	2.00466	2.00591	2.00724	2.00862

Figures

Fig. 1a-b: Exact limit cycle (continuous line) and approximated limit cycle (discontinuous line) up to order $\mathcal{O}(\epsilon^5)$ for the van der Pol system with (a) $\epsilon = 0.8$ and (b) $\epsilon = 2$.

Fig. 2a-b: Exact limit cycle (continuous line) and approximated limit cycle (discontinuous line) up to order $\mathcal{O}(\epsilon^3)$ for the unstable limit cycle of the example 2 with (a) $\epsilon = -0.8$ and (b) $\epsilon = -2$.

Fig. 3a-b: Exact limit cycle (continuous line) and approximated limit cycle (discontinuous line) up to order $\mathcal{O}(\epsilon^3)$ for the stable limit cycle of the example 2 with (a) $\epsilon = 0.4$ and (b) $\epsilon = 0.8$.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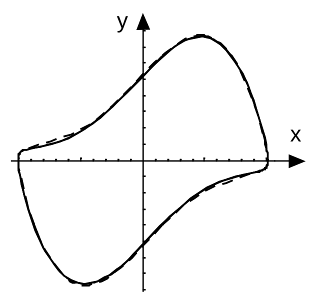
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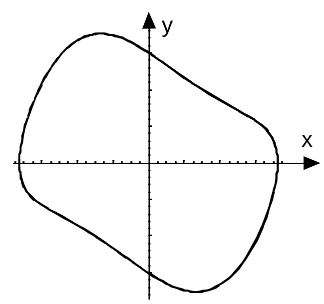
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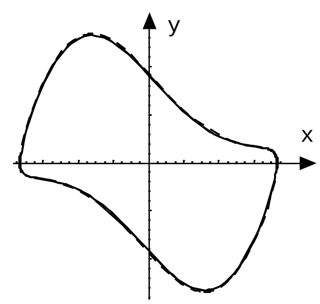
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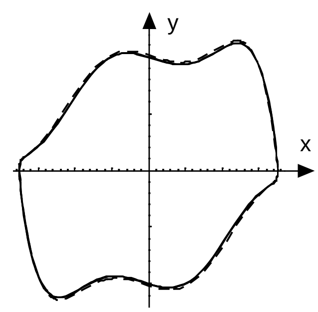
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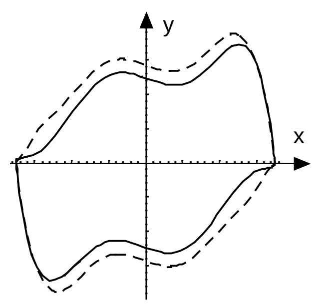
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